# LEBESGUE'S CRITERION FOR RIEMANN-INTEGRABILITY

Background: We have seen earlier, that every continuous function is Riemann integrable. A natural question is — is continuity necessary for a function to be Riemann integrable? If not, then how much amount of discontinuity can we allow for our function? We try to find these answers in this section.

Recall that, if the difference between the upper and lower Riemann sums for a function $f$ which is bounded in $[a,b]$ and having every partition $P$ of $[a,b]$ $\ni \|P\| < \delta$ $(\delta > 0)$ is

$$U(f,P) - L(f,P) < \varepsilon \qquad \forall \varepsilon > 0.$$

$$\Rightarrow \sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta x_k < \varepsilon \qquad \forall \varepsilon > 0.$$

then $f$ will be Riemann integrable. Let us split this sum into 2 parts $S_1 + S_2$ (say) where,

$S_1$ : sum due to the subintervals containg only pts. of continuity of $f$

$S_2$ : sum due to the remaining terms

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Note that, in $S_1$, each $M_k(f)-m_k(f)$ is small due to continuity and hence a large number of such terms can still keep $S_1$ small.

However, in $S_2$, $M_k(f)-m_k(f)$ need not be small.

But since they are bounded by $M$(say), then,

$$|S_2| \leq M \sum_k \Delta x_k$$

So the sum $S_2$ will be small if the sum of the lengths of the subintervals corresponding to $S_2$ is small.

So, what we can really expect is, the set of discontinuities of an integrable function can be covered by intervals whose total length is small. This is the background and the central idea behind the Lebesgue's criterion.

## Some definitions and theorems:

① Null Set / Set with Lebesgue measure zero: A set $A \subset \mathbb{R}$ is called a null set if $\forall \varepsilon > 0$, $\exists$ a sequence $(X_n)$ of intervals $(a_n, b_n)$ (say) such that, $A \subset \bigcup_{n=1}^{\infty} X_n$ and if $\mu(X_n)$ denotes the length of the interval $X_n$, then $\sum_{n=1}^{\infty} \mu(X_n) < \varepsilon$.

(2) Oscillation : Let $f$ be defined and bounded on the interval $S$. If $T \subseteq S$, then we define,

(i) Oscillation of $f$ on $T :=$ $\Omega_f(T) = \sup\{f(x) - f(y) : x \in T, y \in T\}$

(ii) Oscillation of $f$ at $x :=$ $\omega_f(x) = \lim_{h \to 0+} \Omega_f(B(x,h) \cap S)$

Theorem 1 : Let $A_k \subset \mathbb{R}$ are null sets for ~~$k \in \mathbb{R}$~~. $k \in \mathbb{N}$. Then $A = \bigcup_{k=1}^{\infty} A_k$ is also a null set.

Theorem 2 : Let $f$ be defined and bounded on $[a,b]$. Let $\varepsilon > 0$ be given and assume that $\omega_f(x) < \varepsilon \;\; \forall x \in [a,b]$. Then $\exists\, \delta > 0 \ni \forall$ closed subinterval $T \subseteq [a,b]$, we have $\Omega_f(T) < \varepsilon$ whenever the length of $T$ is less than $\delta$.
(i.e $\mu(T) < \delta$)

Theorem 3 : Let $f$ be defined and bounded on $[a,b]$. Let us define, $J_\varepsilon = \{x : x \in [a,b],\ \omega_f(x) \geq \varepsilon\}$ $\forall \varepsilon > 0$. Then $J_\varepsilon$ is a closed set.

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## Lebesgue's criterion for Riemann-integrability:

**Statement:** Let $f$ be defined and bounded on $[a,b]$ and let $A$ denote the set of discontinuities of $f$ in $[a,b]$. Then $f$ is Riemann integrable on $[a,b]$ iff $A$ is a null set (i.e. $A$ has measure zero).

**Proof:** $(\Rightarrow)$

We shall use the method of contradiction to prove this. If possible, let us assume that $A$ does not have measure zero.

We can write $A$ as:

$$A = \bigcup_{r=1}^{\infty} A_r$$

where, $A_r = \{x : \omega_f(x) \geq \frac{1}{r}\}$

Now, if $A$ does not have measure zero, then some of the sets $A_r$ does not have measure zero too. (by Th. 1)

$\therefore \exists\ \varepsilon > 0 \ni$ every countable collection of open intervals covering $A_r$ has a sum of lengths $\geq \varepsilon$.

For any partition $P$ of $[a,b]$,

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} (M_k(f) - m_k(f))\,\Delta x_k = S_1 + S_2 \geq S_1$$

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where, $S_1$ : sum of those terms coming from subintervals containing pts. of A in their interior

$S_2$ : sum of the remaining terms.

Now, the open intervals from $S_1$ cover $A_r$ except possibly for a finite subset of $A_r$ having measure zero.
$\Rightarrow$ Sum of their lengths is atleast $\varepsilon$.

Again, $M_k(f) - m_k(f) \geq \frac{1}{r}$

$$\Rightarrow S_1 \geq \frac{\varepsilon}{r} \quad \Rightarrow U(f,P) - L(f,P) \geq S_1 = \varepsilon/r .$$

for every partition P.

$\therefore$ Then, Riemann condition is not satisfied which is a contradiction.

Hence, our assumption is wrong and A has to be a null set (has measure zero).

($\Leftarrow$)

Now we assume that A has measure zero and show that the Riemann condition is satisfied.

Again, we can express A as :

$$A = \bigcup_{r=1}^{\infty} A_r$$

where, $A_r = \left\{ x : \omega_f(x) \geq \frac{1}{r} \right\}$

$\because A_r \subseteq A \Rightarrow A_r$ has measure zero $\forall r \in \mathbb{N}$. (Th. 1)

$\Rightarrow A_r$ can be covered by open intervals, sum of whose lengths $< \frac{1}{r}$.

Again, since $A_r$ is compact, (Th. 3), a finite number of these intervals cover $A_r$.

Let us denote the union of these intervals by $B_r$.

$\Rightarrow B_r$ is an open set (union of open intervals).

$\Rightarrow C_r = B_r' = [a,b] - B_r \equiv$ Union of finite no. of closed sub-intervals of $[a,b]$.

Now, let $I :=$ a typical subinterval of $C_r$.

$x \in I \Rightarrow \omega_f(x) < \frac{1}{r}$

$\therefore \exists \delta > 0 \ni I$ can be further sub-divided into a finite no. of subintervals $J$ of length $< \delta$, $\ni \Omega_f(J) < \frac{1}{r}$. (Th. 2)

The endpoints of all these subintervals determine a partition $P_r$ of $[a,b]$. If $P$ is finer than $P_r$, then,

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} (M_k(f) - m_k(f))\, \Delta x_k$$

$$= S_1 + S_2$$

Where, $S_1$: sum of those terms coming from subintervals containing pts. of $A_r$

$S_2$: sum of the remaining terms.

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Let the whole process be done in such a way that, we choose $r$, as large such that,

$$\frac{1}{r} < \frac{\varepsilon}{2(b-a)}$$

$\therefore M_k(f) - m_k(f) < \frac{1}{r} \Rightarrow S_2 < \frac{b-a}{r}$

$\Rightarrow S_2 < \varepsilon/2$ — (i)

Again, we choose $B_r$ in such a way that, the net length of the intervals is $< \frac{\varepsilon}{2(M-m)}$

M: sup (f) on [a, b]
m: inf (f) on [a, b].

$\because$ $B_r$ covers all ~~sub~~ intervals contributing to $S_1$, we have,

$$S_1 \leq \frac{M-m}{r}$$

$\Rightarrow S_1 \leq \varepsilon/2$ — (ii)

Now, adding (i) and (ii), $S_1 + S_2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$\Rightarrow U(f, P) - L(f, P) < \varepsilon$

$\therefore$ The Riemann condition is satisfied.

Hence, the proof.

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